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An alternative proof of a theorem of Stieltjes and related results

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Abstract

Let $P_n(x)$, $n \geqslant 1$ be the orthogonal polynomials defined by

$$a_n P_{n+1}(x) + a_{n-1} P_{n-1}(x) + b_n P_n(x) = x P_n(x), \quad P_0(x) = 0, \; P_1(x) = 1,$$

where both sequences a_n and b_n are bounded and $a_n > 0$.

Assume that $\psi(x)$ is the unique (up to a constant) distribution function which corresponds to the measure of orthogonality of $P_n(x)$ and denote by $S(\psi)$ the spectrum of $\psi(x)$. Alternative proofs of a theorem due to Stieltjes and of a conjecture due to Maki concerning the limit points of $S(\psi)$ are given. A typical example to the Maki's conjecture together with a general result concerning the density of the zeros of the polynomials $P_n(x)$ covers as a particular case a theorem of Chihara which generalizes the well-known theorem of Blumenthal.

Keywords: Orthogonal polynomials; Measure of orthogonality; Zeros; Limit points of the spectrum

AMS Classification: 42C05

1. Introduction

In [7] Chihara has formulated and proved the following theorem which follows from a result of Stieltjes [16], concerning continued fractions.

Theorem 1.1 (Stieltjes [16]). *Let R_n be the set of orthogonal polynomials defined by*

$$\begin{aligned} &a_n R_{n+1}(x) + a_{n-1} R_{n-1}(x) = x R_n(x), \quad n \geqslant 1, \\ &R_0(x) = 0, \qquad R_1(x) = 1. \end{aligned} \tag{1.1}$$

Then a necessary and sufficient condition for the associated distribution function ψ to have a denumerable spectrum whose only limit point is zero is $\lim_{n\to\infty} a_n = 0$.

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In this paper we begin with an alternative proof of the more general case of nonsymmetric polynomials

$$a_n P_{n+1}(x) + a_{n-1} P_{n-1}(x) + b_n P_n(x) = x P_n(x),$$
$$P_0(x) = 0, \qquad P_1(x) = 1. \tag{1.2}$$

It is well known that the proof for this general case follows also from a theorem of Krein [2, pp. 230–231]. The proof we follow is based on well-known properties of compact operators. The well-known Weyl's theorem concerning invariance of the essential spectrum by compact perturbations is also used to prove a conjecture of Maki [13], which has been proved by Chihara [8] with a different method. This conjecture asserts that in the case $\lim_{n\to\infty} a_n = 0$ the point λ is a limit point of the spectrum of ψ if and only if it is a limit point of the sequence $\{b_n\}$.

Let $\lim a_n = \gamma$, $\lim b_{2n-1} = \alpha$, $\lim b_{2n} = \beta$.

We prove that in the union of the two closed intervals

$$[-\sqrt{k^2 + 4\gamma^2} + k + \beta, \beta], \qquad [\alpha, \sqrt{k^2 + 4\gamma^2} + k + \beta],$$

where $2k = \alpha - \beta$, the distribution function ψ is continuous, and in particular the points α and β are limit points of the spectrum of ψ. For $\gamma = 0$ this is a typical example to the Maki's conjecture.

This example is also of interest because, together with a general result concerning the density of the zeros of the polynomials, it covers as a particular case a theorem of Chihara [2] which generalizes the well-known theorem of Blumenthal [6], [10, pp. 122–124].

2. Preliminary results

We consider the tridiagonal operator T, which corresponds to the sequences a_n, b_n of (1.2)

$$Te_n = a_n e_{n+1} + a_{n-1} e_{n-1} + b_n e_n, \quad n \geqslant 1,$$
$$Te_1 = a_1 e_2 + b_1 e_1. \tag{2.1}$$

In (2.1) e_n is an orthonormal basis in an abstract separable Hilbert space H. This operator can be separated as

$$T = VA + AV^* + B,$$

where A, B are the diagonal operators $Ae_n = a_n e_n$, $Be_n = b_n e_n$, V is the unilateral shift $Ve_n = e_{n+1}$ and V^* its adjoint ($V^* e_n = e_{n-1}$, $V^* e_1 = 0$).

If the sequences a_n and b_n are bounded the diagonal operators A, B are bounded. Consequently, T is also a bounded operator. Therefore, there is no problem with respect to the self-adjointness of T. Thus T is a bounded self-adjoint operator on H. To this self-adjoint operator there corresponds a unique spectral family of projection operators E_t such that $\psi(t) = (E_t e_1, e_1)$ is the measure of orthogonality of the polynomials $P_n(t)$, i.e.,

$$\int_\alpha^b P_n(t) P_m(t) \, \mathrm{d}(E_t e_1, e_1) = \delta_{n,m},$$

where $\alpha = \inf\{t: t \in \sigma(T) = \text{spectrum of } T\}$, $b = \sup\{t: t \in \sigma(T)\}$. Moreover, the spectrum of the function $\psi(t)$ defined by $S(\psi) = \{t: \psi(t + \delta) - \psi(t - \delta) > 0$, for all $\delta > 0\}$, coincides with the spectrum $\sigma(T)$ of the operator T. Among books, where these subjects and their relationships are treated we mention [1, 4, 15, 17].

The essential spectrum of a self-adjoint operator T ($\sigma_e(T)$) consists of the continuous spectrum of T ($\sigma_c(T)$), the eigenvalues of T with infinite multiplicity and the points which are accumulation points of eigenvalues. Weyl's theorem, see for instance [14], asserts that if T and T_0 are self-adjoint and T_0 is compact then $\sigma_e(T) = \sigma_e(T + T_0)$.

3. Alternative proofs

We shall prove the following theorem.

Theorem 3.1. *Let $P_n(x)$ be the set of polynomials defined by (1.2), where $a_n > 0$ and b_n real. Then a necessary and sufficient condition for the associated distribution function ψ to have a denumerable spectrum whose only limit point is b is*

$$\lim_{n\to\infty} a_n = 0 \quad \textit{and} \quad \lim_{n\to\infty} b_n = b. \tag{3.1}$$

Proof. The spectrum of ψ in this case is the spectrum $\sigma(T)$ of the self-adjoint operator

$$T = AV^* + VA + B. \tag{3.2}$$

Without loss of the generality, we take $b = 0$. Assume that the spectrum of T is denumerable consisting of the points E_n, $n \geqslant 1$, with $\lim_{n\to\infty} E_n = 0$. This means that T has a complete system x_n, $n \geqslant 1$, of eigenelements corresponding to the simple eigenvalues $E_1 > E_2 > \cdots > E_n > \cdots$. Since $E_n \xrightarrow[n\to\infty]{} 0$ the operator T is compact [14]. Also since T is compact and since the sequence e_n, $n \geqslant 1$ converges weakly to zero, the sequence Te_n converges strongly to zero [14], i.e.,

$$\lim_{n\to\infty} \| Te_n \| = 0. \tag{3.3}$$

But $Te_n = a_n e_{n+1} + a_{n-1} + b_n e_n$, so

$$\| Te_n \|^2 = a_n^2 + a_{n-1}^2 + b_n^2, \tag{3.4}$$

and (3.1) follows from (3.4) and (3.3).

Conversely, let (3.1) be satisfied with $b = 0$. Then the diagonal operators A and B are compact, consequently, T is compact and self-adjoint. Thus it has a denumerable set of eigenvalues E_n, $n \geqslant 1$, with $\lim_{n\to\infty} E_n = 0$. Since the eigenvalues of T are simple these eigenvalues are distinct. □

Remark 3.1. A compact operator may have finite eigenvalues. In that case the point zero is an eigenvalue of infinite multiplicity. This is impossible because the eigenvalues of the operator (3.2) for $a_n > 0$ are simple. Thus T has an infinite denumerable set of eigenvalues $E_n \neq 0$ with $\lim_{n\to\infty} E_n = 0$.This means that the distribution function which corresponds to the measure of orthogonality of the polynomials $P_n(x)$ is discrete with an infinite number of points at which it increases.

Theorem 3.2. *Let $\lim_{n\to\infty} a_n = 0$. Then λ is a limit point of the sequence $\{b_n\}_{n=1}^{\infty}$ if and only if λ is a limit point of the spectrum $S(\psi)$ of ψ.*

Proof. Since $\lim_{n\to\infty} a_n = 0$, the operator $T_0 = AV^* + VA$, where $Ae_n = a_n e_n$ is compact. Therefore, according to Weyl's theorem the limit points of the spectrum of $T = B + T_0$, where $Be_n = b_n e_n$, $n \geqslant 1$, coincide with the limit points of the spectrum of B. $\square$

4. A typical example

We consider the case

$$\lim_{n\to\infty} a_n = \gamma, \qquad \lim_{n\to\infty} b_{2n-1} = \alpha, \qquad \lim_{n\to\infty} b_{2n} = \beta. \tag{4.1}$$

We shall prove that the closed set

$$[-\sqrt{k^2 + 4\gamma^2} + k + \beta, \beta] \cup [\alpha, \sqrt{k^2 + 4\gamma^2} + k + \beta], \quad \alpha > \beta, \tag{4.2}$$

where

$$2k = \alpha - \beta, \tag{4.3}$$

belongs to the continuous spectrum of the operator

$$T = AV^* + VA + B \tag{4.4}$$

and, in particular, the points α and β are limit points of the spectrum of T.

Let P_1 and P_2 be the projection operators on the subspaces spanned by the elements $\{e_1, e_3, e_5, \dots\}$ and $\{e_2, e_4, e_6, \dots\}$, respectively. Thus

$$P_1 e_{2n-1} = e_{2n-1}, \qquad P_1 e_{2n} = 0, \qquad P_2 e_{2n-1} = 0, \qquad P_2 e_{2n} = e_{2n},$$

$$P_1 + P_2 = I \quad \text{and} \quad B = BP_1 + BP_2 \quad \text{or} \quad B = (B-\alpha)P_1 + (B-\beta)P_2 + \alpha P_1 + \beta P_2,$$

or $B = K + \beta + (\alpha - \beta)P_1$, where the operator $K = (B-\alpha)P_1 + (B-\beta)P_2$, is compact. Since $VA + AV^* = \gamma(V + V^*) + K_1$, where $K_1 = V(A - \gamma I) + (A - \gamma I)V^*$ is also compact, Weyl's theorem implies that the operators

$$T = VA + AV^* + B$$

and

$$T_0 = \gamma(V + V^*) + (\alpha - \beta)P_1 + \beta \tag{4.5}$$

have the same essential spectrum.

Assume without loss of the generality that $\alpha > \beta$ and consider the spectrum of the operator

$$\gamma(V + V^*) + 2kP_1, \quad 2k = \alpha - \beta. \tag{4.6}$$

First we prove that the spectrum of the operator (4.6) is purely continuous. Since this operator is self-adjoint we have to prove that the point spectrum is empty. Assume that there exists an eigenvalue λ, i.e.

$$\gamma(Vf + V^*f) + 2kP_1 f = \lambda f, \quad f \neq 0. \tag{4.7}$$

We observe that

$$(f, e_1) \neq 0 \tag{4.8}$$

because, otherwise, from (4.7) scalar product multiplication by e_1 implies that $\gamma(f, e_2) + k(P_1 f, e_1) = 0$ or $\gamma(f, e_2) + k(f, e_1) = 0$ and $(f, e_2) = 0$. Consequently, scalar product multiplication by e_2 implies that $(f, e_3) = 0$ and so on. Thus, $(f, e_n) = 0$ for every n, which means $f = 0$, contrary to (4.7). Eq. (4.7) can be transformed into a functional equation in the Hardy–Lebesgue space $H_2(\Delta)$, i.e., the Hilbert space consisting of the analytic functions $f(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1}$ in the open unit disc $\Delta = \{z: |z| < 1\}$ which satisfy $\sum_{n=1}^{\infty} |\alpha_n|^2 < +\infty$. In fact by the representation [11],

$$f(z) = (f_z, f), \qquad f \in H, \qquad f_z = \sum_{n=1}^{\infty} z^{n-1} e_n, \tag{4.9}$$

the eigenvalue problem (4.7) in H is equivalent to the eigenvalue problem of the equation

$$\gamma\left[\frac{1}{z}(f(z) - f(0)) + zf(z)\right] + k(f(z) + f(-z)) = \lambda f(z) \tag{4.10}$$

in $H_2(\Delta)$, i.e., equivalent to the problem of finding the values of λ for which Eq. (4.10) has a solution in $H_2(\Delta)$. Since $k \neq 0$, we find from (4.10) that

$$f(-z) = \frac{(z^2 + \gamma + (k - \lambda)z)f(z) - \gamma f(0)}{kz}. \tag{4.11}$$

If $f(z) \in H_2(\Delta)$ then by symmetry $f(-z) \in H_2(\Delta)$ and from (4.11) we must have $f(0) = (e_1, f) = 0$, contrary to (4.8).

Regular points of the operator $T = \gamma(V + V^*) + 2kP_1$ are the points λ such that the equation

$$\gamma(V + V^*)f + (2kP_1 - \lambda)f = g$$

has a unique solution in H for every $g \in H$. Consequently, the points λ for which the equation

$$\gamma(V + V^*)f + (2kP_1 - \lambda)f = e_1 \tag{4.12}$$

has no solution in H belong to the spectrum. Eq. (4.12) has a solution in H if and only if the following functional equation:

$$\gamma\left[zf(z) + \frac{1}{z}(f(z) - f(0))\right] + k(f(z) + f(-z)) - \lambda z = 1 \tag{4.13}$$

has a solution in $H_2(\Delta)$. From (4.13) we find that

$$f(z)(\gamma z^2 + \gamma + (k - \lambda)z) = \gamma f(0) + z - kzf(-z) \tag{4.14}$$

and

$$f(-z)(\gamma z^2 + \gamma - (k - \lambda)z) = \gamma f(0) - z + kzf(z). \tag{4.15}$$

Elimination of $f(-z)$ gives

$$f(z) = \frac{f(0)[\gamma^2 z^2 + \gamma^2 - 2\gamma kz + \gamma\lambda z] + z(\gamma z^2 + \lambda z + \gamma)}{\gamma^2 z^4 - (\lambda^2 - 2k\lambda - 2\gamma^2)z^2 + \gamma^2}. \tag{4.16}$$

For the points λ for which the inequality

$$(\lambda^2 - 2k\lambda - 2\gamma^2)^2 - 4\gamma^4 = (\lambda^2 - 2k\lambda - 4\gamma^2)(\lambda^2 - 2k\lambda) < 0 \tag{4.17}$$

holds the equation $\gamma^2 z^4 - (\lambda^2 - 2k\lambda - 2\gamma^2)z^2 + \gamma^2 = 0$ has solutions with absolute value equal to 1 and therefore $f(z)$ does not belong to $H_2(\Delta)$. The points λ for which inequality (4.17) holds are the open intervals

$$(-\sqrt{k^2 + 4\gamma^2} + k, 0) \quad \text{and} \quad (2k, k + \sqrt{k^2 + 4\gamma^2}).$$

Thus, the above intervals belong to the spectrum and since the spectrum is a closed set it follows that the end points of these intervals belong also to the spectrum. This result together with the result that the point spectrum is empty proves that the set $[-\sqrt{k^2 + 4\gamma^2} + k, 0] \cup [2k, \sqrt{k^2 + 4\gamma^2} + k]$ belongs to the continuous spectrum of the operator (4.6) and therefore the set (4.2) belongs to the continuous spectrum of the operator (4.5).

5. Density of the zeros

The following result concerns the density of all zeros of all polynomials defined by (1.2).

Theorem 5.1. *If $T = AV^* + VA + B$ is self-adjoint, then every point in the spectrum of T is a limit point of the set of all zeros of all polynomials defined by* (1.2).

Proof. The operator T can be strongly approximated by a sequence of truncated finite-dimensional operators $T_N = P_N T P_N$, where P_N is the projection operator on the subspace spanned by $\{e_1, e_2, \ldots, e_N\}$ i.e.,

$$\lim_{N \to \infty} \| Tf - T_N f \| = 0$$

for every $f \in D(T)$. We prove this as follows. For every $f \in D(T)$ we have

$$\| Tf - P_N T P_N \|^2 = \sum_{n=N+1}^{\infty} |(Tf, e_n)|^2 + a_N^2 |(f, e_{N+1})|^2 = \sum_{n=N+1}^{\infty} |(Tf, e_n)|^2 + |(AV^* f, e_N)|^2. \tag{5.1}$$

The first member on the right-hand side of (5.1) tends to zero as $N \to \infty$ because $\| Tf \|^2 = \sum_{n=1}^{\infty} |(Tf, e_n)|^2 < +\infty$ and the second because $AV^* f \in H$, $\forall f \in D(T)$ and the sequence e_n

converges weakly to zero. In fact, $D(AV^*) \supseteq D(T) = D(AV^*) \cap D(VA) \cap D(B)$ i.e., $\|AV^*f\| < +\infty$, $\forall f \in D(T)$ or $AV^*f \in H$, $\forall f \in D(T)$.

Moreover, the eigenvalues of T_N are the zeros of the polynomial $P_{N+1}(x)$, defined by (1.2) (see [12] for details).

Suppose that $\lambda \in \sigma(T)$ and that λ is not a limit point of the zeros of the polynomials $P_{n+1}(x)$, $n \geqslant 1$, or equivalently λ is not a limit point of eigenvalues of the operators T_n. Then there exists $d > 0$ and a subsequence of the operators T_n which we denote by T_N such that $|\lambda - \varrho| \geqslant d$ for every eigenvalue ϱ belonging to any of the operators T_N, $N = 1, 2, \ldots$.

Let $T_N x_k = \lambda_k x_k$, $k = 1, 2, \ldots, N$, $(x_i, x_j) = \delta_{ij}$.

Then for every $f \in D(T)$ we have

$$(\lambda P_N - P_N T P_N) f = \sum_{k=1}^{N} (\lambda - \lambda_k)(f, x_k) x_k$$

and

$$\|(\lambda P_N - P_N T P_N) f\|^2 = \sum_{k=1}^{N} |\lambda - \lambda_k|^2 |(f, x_k)|^2,$$

which implies that

$$\|\lambda P_N f - P_N T P_N f\| \geqslant d \|P_N f\|, \quad f \in D(T),$$

or

$$\|P_N(\lambda I - T) P_N f\| \geqslant d \|P_N f\|, \quad f \in D(T).$$

Since $P_N T P_N$ converges strongly to T and since $\lim_{N\to\infty} \|P_N f\| = \|f\|$, the last inequality for $N \to \infty$ leads to $\|(\lambda I - T)f\| \geqslant d\|f\|, f \in D(T)$. Since T is self-adjoint this means that λ is a regular point of T, contrary to the assumption. □

Remark 5.1. Recently, in [3, Theorem 2.3] and [5, Theorem 1.2] the authors, among others, prove essentially Theorem 5.1 with different methods and different purposes in mind.

From the above theorem we obtain immediately the following corollary.

Corollary 5.1. *The zeros of all the polynomials defined by* (1.2) *are dense in intervals covering by the continuous spectrum of T.*

This corollary is a general result and covers as particular case the following theorem of Chihara [8, pp. 122–124], which generalizes the well-known theorem of Blumenthal [4].

Theorem 5.2. *The zeros of all the polynomials, defined by*

$$a_n P_{n+1}(x) + a_{n-1} P_{n-1}(x) + b_n P_n(x) = x P_n(x),$$

$$P_0(x) = 0, \qquad P_1(x) = 1,$$

with $\lim_{n\to\infty} a_n = \gamma$, $\lim_{n\to\infty} b_{2n-1} = \alpha$, $\lim_{n\to\infty} b_{2n} = \beta$, *are dense in the union of the closed intervals*

$$[-\sqrt{k^2 + 4\gamma^2} + k + \beta, \beta] \cup [\alpha, \sqrt{k^2 + 4\gamma^2} + k + \beta], \quad 2k = \alpha - \beta, \ \alpha > \beta.$$

Proof. From the typical example in Section 4.1 we have that the union $[-\sqrt{k^2+4\gamma^2}+k+\beta,\beta]\cup[\alpha,\sqrt{k^2+4\gamma^2}+k+\beta]$ belongs to the continuous spectrum of the operator (4.5). The operator $T=AV^*+VA+B$ in that case can be written as follows:

$$T=\gamma(V+V^*)+B+(A-\gamma I)V^*+V(A-\gamma I)=T_0+K, \tag{5.2}$$

where T_0 is the operator (4.5) and K is self-adjoint and compact. Weyl's theorem applied to (5.2) gives that the operators T and T_0 have the same essential spectrum. Since the continuous spectrum is a part of the essential spectrum, Corollary 5.1 proves Theorem 5.2. □

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